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Coupled heat and mass transfer by natural convection from vertical surfaces in porous media

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Abstract—Similarity solutions for buoyancy induced heat and mass transfer from a vertical plate embedded in a saturated porous medium are reported for (1) constant wall temperature and concentration, (2) constant wall heat and mass flux. In addition, the effect of flow injection on the heat and mass transfer has also been considered. Governing parameters for the problem under study are the buoyancy ratio, N, and Lewis number, Le. Depending on the sign of the buoyancy ratio, inclusion of a concentration gradient may either assist or suppress the flow induced by thermal buoyancy. The Lewis number is found to have a more pronounced effect on the concentration field than it does on the flow and temperature fields. Results for Nusselt and Sherwood numbers cover a wide range of the governing parameters, $-1 < N < 10$ and $0.1 \leqslant Le \leqslant 100$, and a comparison is made with the results of Bejan and Khair (*Int. J. Heat Mass Transfer* **28,** 909–918 (1985)).

INTRODUCTION

COUPLED heat and mass transfer due to buoyancy in saturated porous media has many important applications in energy-related engineering problems, for example, the migration of moisture in fibrous insulation, the spreading of chemical pollutants in saturated soil, and the underground disposal of nuclear wastes. From a fundamental perspective, Nield [1] made the first attempt to study the stability of convective flow in horizontal layers with imposed vertical temperature and concentration gradients. This was followed by Khan and Zebib [2] in the study of flow stability in a vertical porous layer. Recently, Bejan and co-workers [3–5] conducted a series of investigations of these effects on natural convection for various geometries. For a vertical plate in a saturated porous medium, Bejan and Khair [3] reported similarity solutions for the special case of a wall with constant temperature and concentration.

The purpose of the present study is to re-examine this fundamental case. In the present study, the solutions cover a wide range of parameters, $-1 < N < 10$ and $0.1 \leqslant Le \leqslant 100$, as well as the important case of a wall with constant heat and mass flux. In addition, we include the effect of wall injection on the heat and mass transfer. It is expected that the solutions thus obtained will have useful applications in practice and will serve as a complement to the existing literature.

ANALYSIS AND PROBLEM FORMULATION

Consider a vertical plate embedded in a saturated porous medium. For heat and mass transfer driven only by buoyancy, the governing equations based on Darcy's law are simplified by introducing the stream function such that

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{Kg}{\nu}\left(\beta_T \frac{\partial T}{\partial y} + \beta_c \frac{\partial c}{\partial y}\right) \tag{1}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{2}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial c}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial c}{\partial y} = D \frac{\partial^2 c}{\partial y^2}. \tag{3}$$

In equations (1)–(3), the boundary-layer and Boussinesq approximations have been invoked. In accordance with the linear Boussinesq approximation, density is given as

$$\rho = \rho_\infty[1 - \beta_T(T - T_\infty) - \beta_c(c - c_\infty)]. \tag{4}$$

Case 1. *No interfacial velocity*

If the transfer process occurs at low concentration difference such that the interfacial velocity due to mass diffusion can be neglected, boundary conditions are

$$y = 0, \quad T_w = T_\infty + Ax^a, \quad c_w = c_\infty + Bx^b,$$
$$v = -\frac{\partial \psi}{\partial x} = 0 \tag{5}$$

$$y \to \infty, \quad T = T_\infty, \quad c = c_\infty, \quad u = \frac{\partial \psi}{\partial y} = 0. \tag{6}$$

Similarity solutions to equations (1)–(6) are obtained via the variable transformations $\eta = Ra^{1/2}\, x/y$ and $\psi = \alpha\, Ra^{1/2}\, F(\eta)$. The resulting equations are

$$F' = \theta + NC' \tag{7}$$

$$\theta'' = aF'\theta - \frac{a+1}{2}F\theta' \tag{8}$$

NOMENCLATURE

A	constant defined in equation (5)
a	constant defined in equation (5)
B	constant defined in equation (5)
b	constant defined in equation (5)
C	dimensionless concentration, $(c-c_\infty)/(c_w-c_\infty)$
c	concentration
D	mass diffusivity
F	dimensionless stream function
f_w	flow injection parameter defined by equation (13)
G	auxiliary function, $\partial F/\partial \xi$
g	acceleration due to gravity
h	local heat transfer coefficient
K	permeability
k	effective thermal conductivity
Le	Lewis number, α/D
m	local mass flux at the wall
N	buoyancy ratio, $[\beta_c(c_w-c_\infty)]/[\beta_T(T_w-T_\infty)]$
Nu	local Nusselt number, hx/k
p	pressure
q	local heat flux at the wall
Ra	modified local Rayleigh number, $Kg\beta_T(T_w-T_\infty)x/\alpha\nu$
S	auxiliary function, $\partial C/\partial \xi$
Sh	local Sherwood number, $mx/D(c_w-c_\infty)$
T	temperature
u	Darcy velocity in the x-direction
v	Darcy velocity in the y-direction
v_w	fluid injection velocity
x, y	Cartesian coordinate.

Greek symbols

α	thermal diffusivity of porous medium
β_T	coefficient of thermal expansion, $(-1/\rho)(\partial\rho/\partial T)_P$
β_c	coefficient of concentration expansion, $(-1/\rho)(\partial\rho/\partial c)_P$
η	independent similarity variable
θ	dimensionless temperature, $(T-T_\infty)/(T_w-T_\infty)$
ν	kinematic viscosity
ξ	non-similarity variable defined in equation (21)
ρ	density of convective fluid
ϕ	auxiliary function, $\partial\theta/\partial\xi$
ψ	stream function.

Subscripts

w	condition at the wall
∞	condition at infinity
0	quantity related to the condition of no interfacial velocity
ξ	quantity related to the case of constant wall injection.

$$C'' = Le\left(bF'C - \frac{a+1}{2}FC'\right) \quad (9)$$

with boundary conditions

$$\eta = 0, \quad \theta = 1, \quad C = 1, \quad F = 0 \quad (10)$$

$$\eta \to \infty, \quad \theta = 0, \quad C = 0, \quad F' = 0. \quad (11)$$

Equation (7) has been simplified by invoking the boundary condition at infinity. The parameter N measures the relative importance of mass and thermal diffusion in the buoyancy-driven flow. It is clear that N is zero for thermal-driven flow, infinite for mass-driven flow, positive for aiding flow, and negative for opposing flow.

It has been shown that similarity solutions exist for the case of thermally-driven flows [6]. Also, it can be shown that similarity solutions can be obtained for the two important cases: (1) $a = b = 0$, which corresponds to a wall with constant temperature and concentration, and (2) $a = b = 1/3$, which corresponds to a wall with constant heat and mass flux. It is important to note that the parallel double boundary-layer structure is a necessary condition for the existence of similarity solutions. For the first case, Bejan and Khair [3] reported similarity solutions for $-5 \leqslant N \leqslant 4$ and $1 \leqslant Le \leqslant 100$, except for $-1 < N < 0$. Solutions for the second case, however, have not apparently been reported thus far.

Case 2. *Effect of flow injection*

In some applications, there may exist an interfacial velocity at the wall due to injection. Two injection velocity profiles are considered here.

For a power-law variation of injection velocity, the governing equations are the same as equations (7)–(9). However, the boundary condition at the wall, equation (5), becomes

$$y = 0,$$

$$v = v_w = v_0 x^n = -\frac{\alpha}{2}(a+1)\left[\frac{Kg\beta_T A}{\alpha\nu}\right]^{1/2} x^{(a-1)/2} F \quad (12)$$

where v_0 is a positive constant. It is obvious that the problem also permits similarity solutions if $n = (a-1)/2$.

In terms of similarity variables, the boundary condition at the wall is

$$\eta = 0, \quad F = f_w = -\frac{2v_0}{(a+1)\alpha\left[\frac{Kg\beta_T A}{\alpha\nu}\right]^{1/2}}. \quad (13)$$

For constant injection velocity, the problem does not permit similarity solutions. In this case, the local non-similarity method [7–10] is employed instead. Equations (7)–(9) are transformed into a set of six simultaneous equations given by

$$F' = \theta + NC \tag{14}$$

$$\theta'' = aF'\theta - \frac{a+1}{2}F\theta' + \frac{a-1}{2}\xi(G\theta' - F'\phi) \tag{15}$$

$$C'' = Le\left[bF'C - \frac{a+1}{2}FC' + \frac{a-1}{2}\xi(GC' - F'S)\right] \tag{16}$$

$$G' = \phi + NS \tag{17}$$

$$\phi'' = aG'\theta - G\theta' + \frac{a+1}{2}(F'\phi - F\phi') + \frac{a-1}{2}\xi(G\phi' - G'\phi) \tag{18}$$

$$S'' = Le\left[bG'C - GC' + \left(b - \frac{a-1}{2}\right)F'S - \frac{a+1}{2}FS' + \frac{a-1}{2}\xi(GS' - G'S)\right] \tag{19}$$

with boundary conditions

$$\eta = 0, \quad \theta = 1, \quad C = 1, \quad F = -\frac{2\xi}{a+1}\left(1 - \frac{a-1}{2}G\right)$$
$$\phi = 0, \quad S = 0, \quad G = -1 \tag{20a}$$

$$\eta \to \infty, \quad \theta = 0, \quad C = 0, \quad F' = 1$$
$$\phi = 0, \quad S = 0, \quad G' = 0. \tag{20b}$$

The non-similarity variable, ξ, is defined by

$$\xi = \frac{v_w}{\alpha}\left[\frac{\nu\alpha}{Kg\beta_T A}\right]^{1/2} x^{-(a-1)/2} \tag{21}$$

while G, ϕ and S are the auxiliary functions.

RESULTS

The transformed ordinary differential equations, with the corresponding boundary conditions, are solved numerically using the fourth-order Runge–Kutta method and the shooting technique with a systematic guessing of $\theta'(0)$ and $C'(0)$. To satisfy boundary conditions at infinity, the integration length needs to be chosen carefully and depends on the Lewis number and buoyancy ratio. For our calculations, it varies from 10 to 40 for satisfactory solutions. As an indication of proper formulation and accuracy, the results thus obtained have been compared with the published data and show excellent agreement.

Results of practical interest are heat and mass transfer rates, which are computed from

$$q = -k\frac{\partial T}{\partial y}\bigg|_{y=0} = -kA\left(\frac{Kg\beta_T A}{\nu\alpha}\right)^{1/2} x^{(3a-1)/2}\theta'(0) \tag{22}$$

$$m = -D\frac{\partial c}{\partial y}\bigg|_{y=0} = -DB\left(\frac{Kg\beta_T A}{\nu\alpha}\right)^{1/2} x^{(a+2b-1)/2}C'(0). \tag{23}$$

Equations (22) and (23) clearly show that heat and mass flux are constant for $a = b = 1/3$. The heat transfer coefficient in terms of the Nusselt number is given by

$$Nu = \frac{hx}{k} = -Ra^{1/2}\,\theta'(0) \tag{24}$$

and the mass transfer coefficient in terms of the Sherwood number is given by

$$Sh = \frac{mx}{D(c_w - c_\infty)} = -Ra^{1/2}\,C'(0). \tag{25}$$

Equation (24) is plotted in Figs. 1 and 2 as a function of the Lewis number and buoyancy ratio, respectively. For $N > 0$, heat transfer is greatly enhanced by mass buoyancy, while it is considerably reduced for $N < 0$ (Fig. 1). For $N > 1$, the variation of the heat transfer coefficient essentially follows the scale analysis presented by Bejan and Khair [3]. For $Le < 1$, the

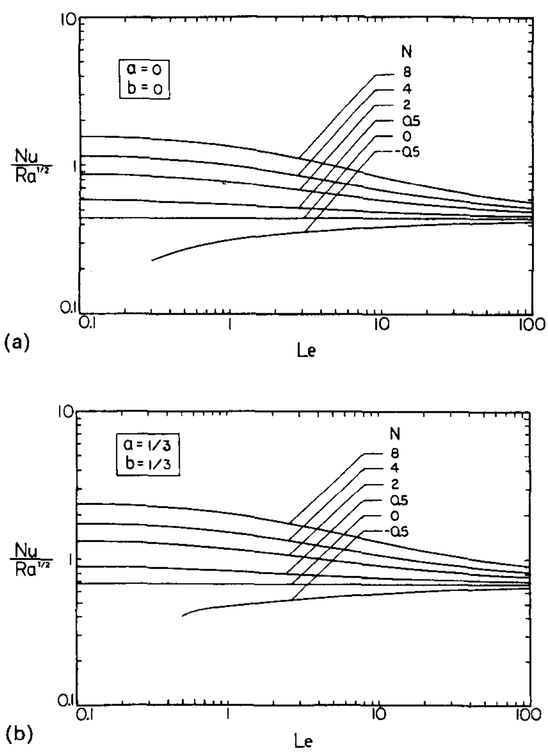

FIG. 1. Heat transfer coefficients as a function of the Lewis number.

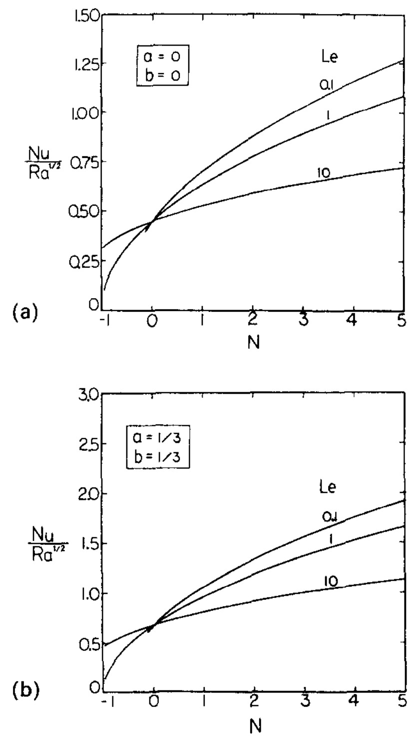

FIG. 2. Heat transfer coefficients as a function of the buoyancy ratio.

heat transfer coefficient is independent of the Lewis number, while it shows a dependence of $Le^{1/2}$ when $Le \to 100$. For $Le < 1$, heat transfer coefficients increase for $N > 0$ and decrease for $N < 0$ (Fig. 2). For $Le > 1$, the situation is reversed.

Contrary to what has been reported by Bejan and Khair [3], we have found solutions for $Le = 1$ and $-1 < N < 0$, and that solutions in the range of $N < -1$ are impossible. These contradictions can be resolved with the aid of equation (7), from which the vertical velocity component can be computed. It is important to note that this velocity component should be positive for the problem under consideration and it is clear that this velocity component will become negative, which means the buoyant flow will be reversed, when $Le = 1$ and $N < -1$. For $Le > 1$, the contribution to the vertical velocity by mass buoyancy is less important, and the flow reversal will occur at a smaller value of N, i.e. $N < -1$. For $Le < 1$, the situation is reversed and flow reversal will take place at a larger value of N (Fig. 2). In all cases, as soon as the flow is reversed, the boundary-layer assumption breaks down, and no solution is meaningful.†

† Although Bejan and Khair [3] did not state clearly in their paper, the solutions they presented for the range of $-5 \leqslant N \leqslant -1$ actually corresponded to a different problem, for which the convective flow is always downward, such that the parallel double boundary-layer structure is maintained.

For mass transfer, equation (25) is plotted in Figs. 3 and 4 as a function of Lewis number and buoyancy ratio, respectively. It is important to note that the mass transfer coefficient does not have a physical significant meaning for thermally-driven flow, i.e. $N = 0$. The curve for $N = 0$ is included in Fig. 3 only for comparison. The variation of the mass transfer coefficient also follows the scale analysis presented by Bejan and Khair [3]. For $N > 1$, the slope of these curves clearly indicates the dependence of the Lewis number to the 1/2-power (Fig. 3). It is also observed that the Lewis number has a more pronounced effect on the concentration field than it does on the temperature field (Fig. 4). The mass transfer coefficient is significantly increased when the Lewis number increases.

When taking into account the effect of flow injection, the heat and mass transfer results are shown in Fig. 5 as a function of the flow injection parameter, f_w. As is the case of thermally-driven flow, injection of fluid at the wall leads to a thicker thermal boundary layer such that the heat transfer coefficient is decreased. For a large value of N, the reduction of the heat transfer coefficient is almost linear with f_w. With constant flow injection, heat and mass transfer coefficients defined above are modified as

$$\frac{Nu}{Ra^{1/2}} = -\theta'(\xi, 0), \quad \frac{Sh}{Ra^{1/2}} = -C'(\xi, 0). \quad (26)$$

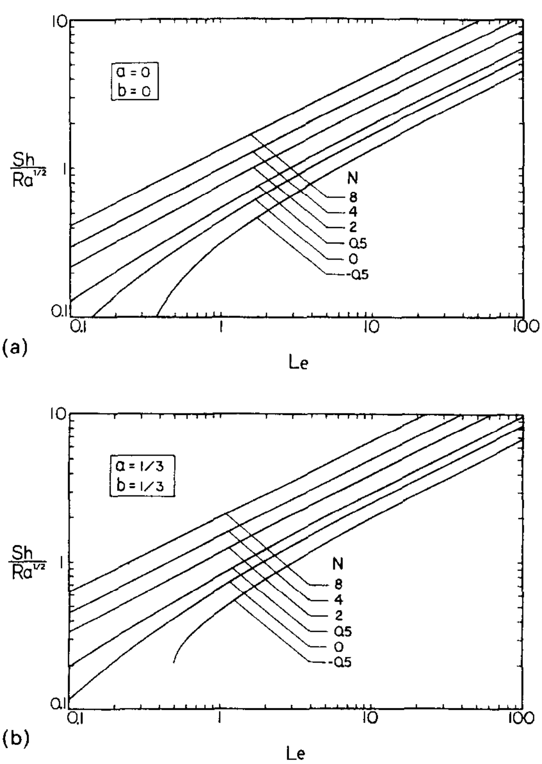

FIG. 3. Mass transfer coefficients as a function of the Lewis number.

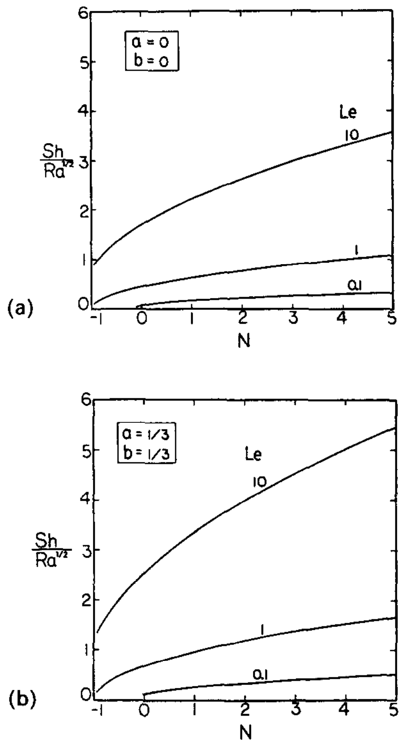

FIG. 4. Mass transfer coefficients as a function of the buoyancy ratio.

The results can be best presented by the ratio of the heat (mass) transfer coefficient for the present case to that for the case of no interfacial velocity (Fig. 6). As the interfacial velocity increases, i.e. ξ increases, the thermal and concentration boundary layers become thicker. As a result, the heat and mass transfer coefficients decrease considerably. The reduction in the heat and mass transfer coefficient is especially significant for $N < 0$.

CONCLUDING REMARKS

Heat and mass transfer coefficients obtained here agree very well with the prediction by scale analysis in a previous study [3]. However, it is also pointed out that the solutions reported in this earlier study for the range of $-5 \leqslant N \leqslant -1$ actually correspond to a downward buoyant flow problem. For the specific problem under study, the boundary-layer structure is destroyed, which occurs, for example, at $Le = 1$ and $N < -1$. When taking into account the effect of wall injection, it is found that the heat and mass transfer coefficients are reduced considerably, especially when $N < 0$.

Problems of the present kind may be encountered in many occasions, for example, in geophysical and geothermal applications where surface mass transfer on the bed (hot) rock is generated due to chemical reaction and in the underground disposal of nuclear

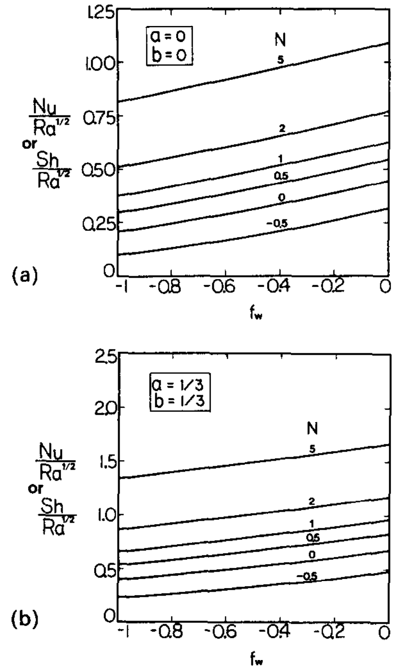

FIG. 5. Effects of power-law flow injection on the heat and mass transfer coefficients, $Le = 1$.

wastes where the spread of radioactive materials may result from the failure of canisters. Given the properties of active species, the present analysis provides an estimation of the life span of a viable geothermal reservoir, or the travel time required for radionuclides to reach the biosphere.

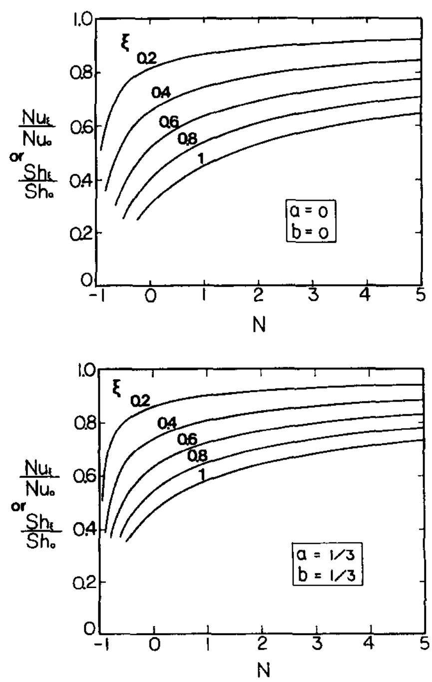

FIG. 6. Effects of constant flow injection on the heat and mass transfer coefficients, $Le = 1$.

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TRANSFERT COUPLE DE CHALEUR ET DE MASSE PAR CONVECTION NATURELLE A PARTIR DE SURFACES VERTICALES DANS UN MILIEU POREUX

Résumé—Des solutions affines pour le transfert de chaleur et de masse par convection naturelle à partir d'une plaque verticale noyée dans un milieu poreux saturé sont rapportées pour (1) température et concentration pariétales constantes, (2) flux pariétal de chaleur et de masse constants. On considère aussi l'effet de l'injection d'écoulement sur le transfert de chaleur et de masse. Les paramètres opératoires sont le rapport de flottement N et le nombre de Lewis Le. Suivant le signe du rapport de flottement, l'inclusion d'un gradient de concentration peut soit aider, soit supprimer l'écoulement induit par le flottement thermique. Le nombre de Lewis a un effet plus prononcé sur le champ de concentration que sur les champs de vitesse et de température. Les résultats des nombres de Nusselt et de Sherwood couvrent un large domaine des paramètres opératoires $-1 < N < 10$ et $0{,}1 \leqslant Le \leqslant 100$, et une comparaison est faite avec les résultats de Bejan et Khair (*Int. J. Heat Mass Transfer* **28,** 909–918 (1985)).

GEKOPPELTER WÄRME- UND STOFFTRANSPORT DURCH NATÜRLICHE KONVEKTION AN SENKRECHTEN OBERFLÄCHEN IN PORÖSEN MEDIEN

Zusammenfassung—Für den Wärme- und Stofftransport durch natürliche Konvektion an einer senkrechten Oberfläche, welche in ein gesättigtes poröses Medium eingebettet ist, werden Ähnlichkeitslösungen für zwei Randbedingungen vorgestellt: (1) Konstante Temperatur und Konzentration an der Wand, (2) konstante Wärme- und Massenstromdichte an der Wand. Zusätzlich wird der Einfluß einer Strömungszufuhr auf den Wärme- und Stofftransport betrachtet. Die bestimmenden Parameter für das untersuchte Problem sind das Auftriebsverhältnis (N) und die Lewis-Zahl (Le). Abhängig vom Vorzeichen des Auftriebsverhältnisses kann die Überlagerung eines Konzentrationsgradienten die thermisch erzeugte Auftriebsströmung entweder unterstützen oder unterdrücken. Es zeigt sich, daß der Einfluß der Lewis-Zahl auf die Konzentrationsverteilung stärker ausgeprägt ist als der Einfluß auf die Geschwindigkeits- und die Temperaturverteilung. In einem großen Bereich der Parameter ($-1 < N < 10$ und $0{,}1 \leqslant Le \leqslant 100$) werden Ergebnisse in Gestalt der Nusselt- und der Sherwood-Zahl ermittelt. Diese Ergebnisse werden mit Resultaten von Bejan und Khair (*Int. J. Heat Mass Transfer* **28**, 909–918 (1985)) verglichen.

ВЗАИМОСВЯЗАННЫЙ СВОБОДНОКОНВЕКТИВНЫЙ ТЕПЛО- И МАССОПЕРЕНОС ОТ ВЕРТИКАЛЬНЫХ ПОВЕРХНОСТЕЙ, ПОГРУЖЕННЫХ В ПОРИСТЫЕ СРЕДЫ

Аннотация—Приводятся автомодельные решения для свободноконвективного тепло- и массопереноса, вызванного погруженной в насыщенную пористую среду вертикальной пластиной для случаев (1) с постоянными температурой стенки и концентрацией, (2) с постоянными тепловым и массовым потоками на стенке. Исследуется также влияние вдува на тепло- и массоперенос, при этом определяющими параметрами в данной задаче являются отношение подъемных сил N и число Льюиса Le. В зависимости от знака отношения подъемных сил наличие градиента концентрации может или способствовать вызванному конвекцией течению, или подавлять его. Найдено, что число Льюиса оказывает более существенное влияние на поле концентраций, чем на поля скоростей и температур. Получены результаты для широкого диапазона изменения чисел Нуссельта и Шервуда, $-1 < N < 10$ и $0{,}1 \leqslant Le \leqslant 100$. Проведено сравнение с результатами Беджана и Кхайра (*Int. J. Heat Mass Transfer* 28, 909–918 (1985)).